

Introduction to Reinforcement Learning

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What is Reinforcement Learning?

- Deep learning is usually presented in the supervised learning setting:
 - ① We have access to labelled training data
 - ② Agent only sees “real world” once it’s already performing well
 - ③ Data is i.i.d between batches
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- RL is vastly different: Agent takes actions in an interactive environment, receive scalar reward as feedback. This lends itself to several problems:
 - ① **Sparse reward:** Sparse feedback to do good
 - ② **Reward attribution:** Which action mattered?
 - ③ **No ground truth:** Optimal policy unknown
 - ④ **Explore vs. Exploit tradeoff:**
 - ① **Exploration:** Learn how the world works
 - ② **Exploitation:** Exploit best understanding to get reward
 - ⑤ **(often) Online only:** No clear distinction between training and testing.

How Much Information is the Machine Given during Learning?

► “Pure” Reinforcement Learning (**cherry**)

- The machine predicts a scalar reward given once in a while.

► **A few bits for some samples**

► Supervised Learning (**icing**)

- The machine predicts a category or a few numbers for each input

► Predicting human-supplied data

► **10→10,000 bits per sample**

► Self-Supervised Learning (**cake génoise**)

- The machine predicts any part of its input for any observed part.

► Predicts future frames in videos

► **Millions of bits per sample**



How Much **(cherry)** is the Machine Given during Learning?

- ▶ “Pure” **(cherry)** Learning **(cherry)**
 - ▶ The machine predicts a scalar reward given once in a while.
 - ▶ Millions of **(cherry)** samples

- ▶ Supervised Learning (apple)
 - ▶ The machine predicts a scalar reward from a few numbers
 - ▶ Millions of samples
 - ▶ 10,000 bits per sample

- ▶ Self-supervised Learning (cake exercise)
 - ▶ The machine predicts a scalar reward from a few numbers
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Agent-Environment Interaction Loop (MDPs)

- MDP is 4-tuple $(\mathcal{S}, T, \mathcal{A}, R)$:
 - *states* $\mathcal{S}$
 - *actions* $\mathcal{A}$
 - *transition distribution* $T : \mathcal{S} \times \mathcal{A} \rightarrow \Delta\mathcal{S}$.
 $T(s'|s, a) = \Pr(s_{t+1} = s' | s_t = s, a_t = a)$.
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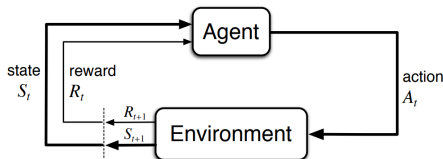
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- On timestep t ,
 - Agent samples $a_t \sim \pi(\cdot | s_t)$, gives action a_t to environment.
 - Environment samples $s_{t+1} \sim T(\cdot | s_t, a_t)$, computes $r_{t+1} = R(s_t, a_t, s_{t+1})$, and gives *percept* (s_{t+1}, r_{t+1}) to agent.

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- Generates an interaction history, or *trajectory*

$s_0, a_0, r_1, s_1, a_1, r_2, s_2, a_2, r_3, s_3, \dots$



Objective of the Agent

- At timestep t , the *return* G_t is the sum of all future rewards:

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- Solutions:**

- Add a discount factor $\gamma \in [0, 1)$ so rewards more imminent are worth more, and the return is always well defined.

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots$$

- Want agent to choose actions to maximise the *expected* return.

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- ❸ **Fully Observable:** The state is a full description of the world
- ❹ **Reward Hypothesis:**

“That all of what we mean by goals and purposes can be well thought of as the maximization of the expected value of the cumulative sum of a received scalar signal (called reward).” -Rich Sutton

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Value Function

$$V_{\pi}(s) = \mathbb{E}_{\pi}[G_t | s_t = s]$$

(Expectation is also with respect to the transitional distribution T .)

Bellman Equation

We note that since

$$\begin{aligned} G_t &= r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots \\ &= r_{t+1} + \gamma(r_{t+2} + \gamma r_{t+3} + \dots) \\ &= r_{t+1} + \gamma G_{t+1} \end{aligned}$$

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$$V_{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s' \in \mathcal{S}} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi}(s'))$$

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- Computing V_{π} from π is called **policy evaluation**, denote $\pi \xrightarrow{E} V_{\pi}$.

Bellman Equation (stochastic policy)

$$V_{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s' \in \mathcal{S}} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi}(s'))$$

Assuming policy $\pi : S \rightarrow A$ is deterministic

Bellman Equation (deterministic policy)

$$V_{\pi}(s) = \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi}(s'))$$

where $a = \pi(s)$.

Bellman Equation (stochastic policy)

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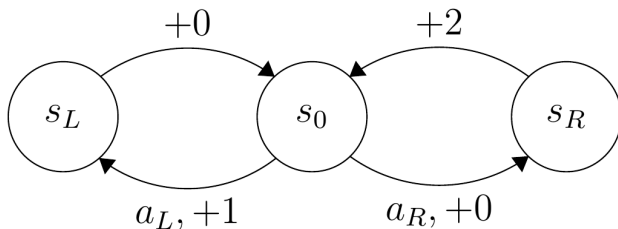
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Only need to sum over all states to find equation for $V_{\pi}(s)$ in terms of the value of all states $\{V_{\pi}(s')\}_{s' \in \mathcal{S}}$.

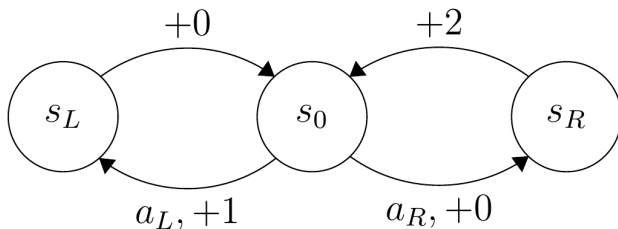
Example Environment

- States $\mathcal{S} = s_0, s_L, s_R$, actions $\mathcal{A} = \{a_L, a_R\}$, possible rewards $\mathcal{R} = \{0, 1, 2\}$.
- Each transition indicates if an action is taken, the reward returned and which state to transition to
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Exercise: Depends on the choice of γ ! Verify that

$$V_{\pi_L}(s_0) = \frac{1}{1 - \gamma^2} \quad V_{\pi_R}(s_0) = \frac{2\gamma}{1 - \gamma^2}$$

so breakeven is at $\gamma = \frac{1}{2}$.

Optimal Bellman

- Policy π_1 is **better** than π_2 ($\pi_1 \geq \pi_2$) if $\forall s. V_{\pi_1}(s) \geq V_{\pi_2}(s)$. A policy is **optimal** if it is better than all other policies.

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Optimal Bellman Equation

$$V_*(s) = \max_a \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_*(s'))$$

Gives a set of **non-linear** simultaneous equations with variables $V_*(s_1), V_*(s_2), \dots$

Problem: No clear way to solve for $V_*(\cdot)$

Can't just compute V_{π} using policy evaluation for all π , as there are $|\mathcal{A}|^{|\mathcal{S}|}$ many to choose from.

Policy Improvement

- Obviously we have that

$$V_*(s) = \max_a \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_*(s')) \quad (1)$$

$$\geq \max_a \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_\pi(s')) \quad (2)$$

$$\geq \sum_{s'} T(s'|s, \pi(s)) (R(s, \pi(s), s') + \gamma V_\pi(s')) \quad (3)$$

$$= V_\pi(s) \quad (4)$$

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Policy Improvement

$$\pi'(s) := \arg \max_a \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_\pi(s'))$$

Exercise: Prove $\pi' \geq \pi$.

Policy Iteration

Policy Improvement (I)

$$\pi'(s) \leftarrow \arg \max_a \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi}(s'))$$

Policy Evaluation (E)

Solve

$$V_{\pi}(s) = \sum_a \pi(a|s) \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi}(s'))$$

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- Start with arbitrary policy π_0 .
- Note that π_* is fixed point of policy improvement.
- Alternate until policy is stable

$$\pi_0 \xrightarrow{E} V_{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} V_{\pi_1} \xrightarrow{E} \pi_2 \xrightarrow{I} V_{\pi_2} \xrightarrow{E} \dots \xrightarrow{I} \pi_* \xrightarrow{E} V_{\pi_*} \xrightarrow{I} \pi_*$$

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Theorem: Policy iteration converges to optimal policy in finitely many steps!

Slippery Gridworld

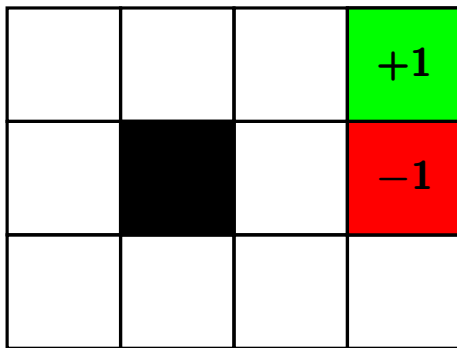
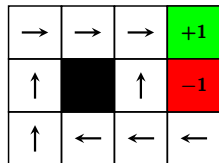
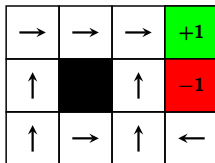


Figure: The slippery gridworld environment. The agent can move in any of the four cardinal directions. Receives reward (punishment?) $\epsilon < 0$ on each timestep. Movement is noisy, with a 70% chance of moving in the desired direction, and 10% of moving in the other three directions. Episode ends when the agent lands in **heaven** **+1** (and gets reward +1) or in **hell** **-1** (and gets reward -1). The wall **■** is impassible, any movement that would walk into the wall, or off the grid, has no effect.

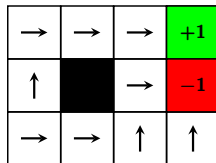
Optimal Policy for Slippery Gridworld



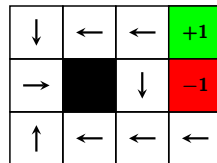
(a) $\epsilon = -0.01$



(b) $\epsilon = -0.1$

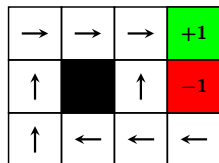


(c) $\epsilon = -2$

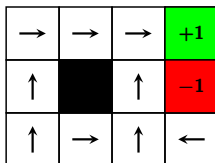


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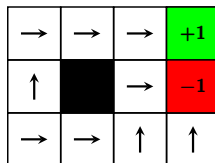
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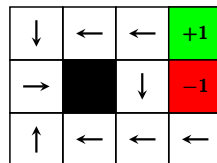
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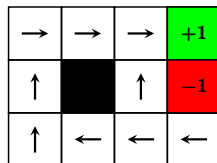
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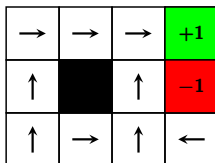
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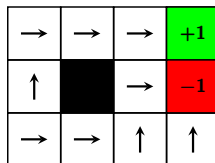
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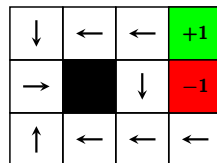
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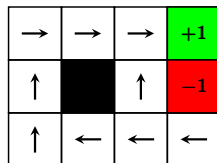
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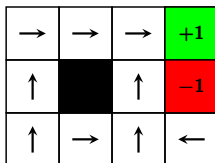
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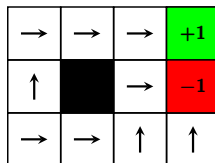
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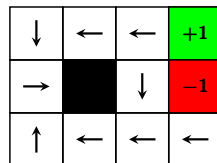
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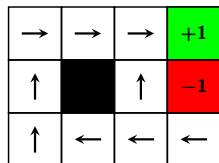
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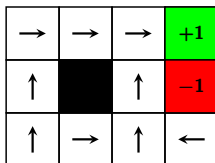
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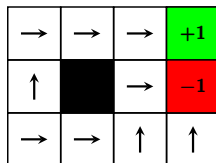
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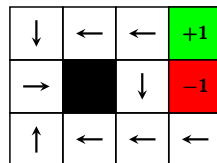
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- (d) **Bliss:** The reward is positive on each timestep, so the agent runs away from heaven or hell to drag out the episode as long as possible.

- **Transition** distribution $T : \mathcal{S} \times \mathcal{A} \rightarrow \Delta\mathcal{S}$, sample $s' \sim T(\cdot|s, a)$.
- **Reward** function $R : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow \mathbb{R}$, write **reward** $r_{t+1} := R(s_t, a_t, s_{t+1})$
- **Policy** $\pi : \mathcal{S} \rightarrow \Delta\mathcal{A}$ (stochastic) or $\pi : \mathcal{S} \rightarrow \mathcal{A}$ (deterministic), take $a_t = \pi(s_t)$ or sample $a_t \sim \pi(\cdot|s_t)$.
- **History/Trajectory** $s_0, a_0, r_1, s_1, a_1, r_2, s_2, a_2, r_3, s_3, \dots$
- **Agent** is the algorithm that selects the *policy* based on the *history*.
- **Discount** $\gamma \in [0, 1)$. Adjusts how far-sighted agent is. $\gamma = 0$ is *myopic*. $\gamma \rightarrow 1$ is *far-sighted*.
- **Return** $G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots = r_{t+1} + \gamma G_{t+1}$ is sum of *future discounted rewards*.
- **Goal** is to maximise G_t .
- **Value** function $V_\pi(s) = \mathbb{E}_\pi[G_t|s_t = s]$.
- **Bellman** equation $V_\pi(s) = \sum_{s' \in \mathcal{S}} T(s'|s, a) (R(s, a, s') + \gamma V_\pi(s'))$
- **Policy Improvement** $\pi'(s) \leftarrow \arg \max_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} T(s'|s, a) (R(s, a, s') + \gamma V_\pi(s'))$
- **Policy Evaluation** Solve $V_\pi(s) = \sum_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} T(s'|s, a) (r + \gamma V_\pi(s'))$ for $V_\pi(s_1), V_\pi(s_2), \dots$

ΔX is the set of all probability vectors $[p_1, \dots, p_n]$ over outcomes in X .

Problems with Policy Iteration

Policy Improvement (I)

$$\pi_{n+1}(s) \leftarrow \arg \max_a \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi_n}(s'))$$

Policy Evaluation (E)

Solve

$$V_{\pi}(s) = \sum_a \pi(a|s) \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi}(s'))$$

for $V_{\pi}(s_1), V_{\pi}(s_2), \dots$

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For the moment, we weaken only the first assumption. Environment is black box, sample percepts (s', r) given state-action pair (s, a) from unknown T and R .

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This then gives us an update rule to improve on our estimate $\hat{V}_{\pi}$ of V_{π} , similar to SGD, called $TD(0)$.

$$\hat{V}_{\pi}(s_t) \leftarrow \hat{V}_{\pi}(s_t) + \alpha \delta_t \equiv \hat{V}_{\pi}(s_t) + \alpha (r_{t+1} + \gamma \hat{V}_{\pi}(s_{t+1}) - \hat{V}_{\pi}(s_t))$$

where $\alpha \in (0, 1]$ is the **learning rate**.

Theorem: Given $\alpha \rightarrow 0$ in a nice way, $TD(0)$ guaranteed to converge to V_{π} .

Q-Value

- **Q-value** is the expected return from state s , taking action a , and thereafter following policy π .

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Q-value Bellman

$$Q_{\pi}(s, a) = \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma Q_{\pi}(s', a'))$$

where $a' = \pi(s')$

Optimal Q-value Bellman

$$Q_{*}(s, a) = \sum_{s'} T(s'|s, a) \left(R(s, a, s') + \max_{a'} Q_{*}(s', a') \right)$$

Q-value vs. Value

Exercise: Can state Q in terms of V , and vice-versa.

$$Q_{\pi}(s, a) = \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_{\pi}(s'))$$

$$V_{\pi}(s) = \sum_{s'} T(s'|s, \pi(s)) (R(s, a, s') + \gamma Q_{\pi}(s', \pi(s')))$$

$$Q_{*}(s, a) = \sum_{s'} T(s'|s, a) (R(s, a, s') + \gamma V_{*}(s'))$$

$$V_{*}(s) = \max_a \sum_{s'} T(s'|s, a) \left(R(s, a, s') + \gamma \max_{a'} Q_{*}(s', a') \right)$$

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- **Idea:** Learn Q_* instead, argmax to recover policy π_*

SARSA: On-Policy TD Control

Apply same argument as TD(0) to the Q-Value

$$Q_*(s, a) = \mathbb{E}_{s' \sim T(\cdot|s,a)} [R(s, a, s') + \gamma Q_*(s', \pi_*(s'))]$$

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Theorem: Under “niceness” conditions SARSA guaranteed to converge to Q_* . 

Q-Learning: Off-Policy TD Control

- Why learn from a_{t+1} when it was a random exploration action? Why not instead learn from the best¹ action $\arg \max_{a'} Q(s_{t+1}, a')$ that should have been taken?

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- Q-Learning (usually) tends to converge faster than SARSA, and chooses more aggressive/risky moves
- SARSA learns from the moves that were actually taken, including any exploration
- In “risky” environments, SARSA will learn to avoid getting near dangerous situations (to avoid accidentally taking a very bad exploratory move). Q-Learning will not.